MMath II - Semestral Examination - Topology III

Time : 3 Hours Max. Marks : 60

Answer all questions. Give complete justifications.

(1) Give example(s) of
  (a) a map $f : S^2 \longrightarrow S^1 \vee S^1 \vee S^2$ that is not null homotopic.
  (b) spaces X, Y and maps $f, g : X \longrightarrow Y$ such that $f_* = g_* : H_i(X;\mathbb{Z}) \longrightarrow H_i(Y;\mathbb{Z})$ for all i but $f_* \neq g_* : H_i(X;G) \longrightarrow H_i(Y;G)$ for some i and some coefficient group G.
  (c) of spaces X and Y that have isomorphic integral homology groups but are not homotopy equivalent. [4+4+4]

(2) A CW-complex X has a cell decomposition with two 0-cells, one 1-cell, and one 3-cell.
  (a) Is X necessarily path connected? [2]
  (b) What can you say about the homology groups of X and the fundamental group $\pi_1(X)$? [5]
  (c) Try to classify such X up to homotopy equivalence by exhibiting as many distinct homotopy classes as you can. [5]

(3) Compute the homology groups of $\mathbb{R}P^2 \times S^2$ with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients. [12]

(4) Let $X = S^2/\{p, q\}$ be the space obtained from the 2-sphere S^2 by identifying two distinct points $p, q \in S^2$. Compute the local homology groups $H_i(X, X - x;\mathbb{Z})$ when $x = [p]$ is the image of the point p (or q) in X and when x is any other point. Hence deduce that any homeomorphism $f : X \longrightarrow X$ must have a fixed point. [8+4]

(5) Let X be a compact connected orientable n-manifold without boundary and suppose that there exists a map $f : S^n \longrightarrow X$ such that the induced homomorphism $f_* : H_n(S^n;\mathbb{Z}) \longrightarrow H_n(X;\mathbb{Z})$ is non trivial. Compute $H^i(X;\mathbb{Q})$ for all $i \geq 0$. You may assume that the integral homology groups of a compact manifold are finitely generated. [12]